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# A method of asymptotic expansions of the solutions of the steady heat conduction problem for laminated non-uniform anisotropic plates☆

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ABSTRACT

Outer asymptotic expansions of the solutions of the steady heat conduction problem for laminated anisotropic non-uniform plates for different boundary conditions on the faces are constructed. The two-dimensional resolvents obtained are analysed and the asymptotic properties of the solutions of the heat-conduction problem are investigated. Estimates are obtained of the accuracy with which the temperature in the plate outside the limits of the boundary layer can be assumed to be piecewise-linearly or piecewise-quadratically distributed over the thickness of the laminated structure. A physical justification for certain features of the asymptotic expansions of the temperature is given.



In an extension of the investigations[1,2] devoted to constructing asymptotic expansions of the solution of the heat conduction problem for thin single-layer plates with respect to a small parameter, the ratio of the thickness of the structure to its characteristic dimension in plan, in this paper we construct asymptotic expansions of the solutions of the steady heat conduction problem for laminated anisotropic plates. This solution, in particular, in thin-walled laminated structures outside the limits of the boundary layer, enables one to estimate with what accuracy the temperature can be specified to be constant or distributed linearly, quadratically or in other forms over the thickness of the plate and its layers.

## 1. Formulation of the heat conduction problem for laminated plates

We will consider a plate of constant thickness $\bar{H}$, consisting of M anisotropic non-uniform layers, also of constant thickness. We will connect with the plate a rectangular Cartesian system of coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$ such that the reference plane $\bar{x}_3 = 0$ coincides with the lower face plane of the plate. We will number all the layers in succession from bottom to top, i.e., the first layer will be the lowest one, while the $M$-th layer will be the upper one. The conditions for ideal thermal contact are satisfied at the boundaries between the layers.

With these assumptions, the equation of steady heat conduction and the relations of Fourier's law for the $m$-th layer have the form

$$(\bar{x}_1, \bar{x}_2) \in \bar{G}, \quad \bar{H}_{m-1} \le \bar{x}_3 \le \bar{H}_m: \quad \sum_{j=1}^{3} \frac{\partial \bar{q}_j^m(\bar{\mathbf{x}})}{\partial \bar{x}_j} = \bar{Q}^m(\bar{\mathbf{x}})$$

$$\bar{q}_i^m(\bar{\mathbf{x}}) = -\sum_{j=1}^{3} \bar{\lambda}_{ij}^m(\bar{\mathbf{x}}) \frac{\partial \bar{T}^m(\bar{\mathbf{x}})}{\partial \bar{x}_j}, \quad i = 1, 2, 3; \quad 1 \le m \le M \tag{1.1}$$

where $\bar{\mathbf{x}} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, $\bar{T}^m$ is the temperature of the $m$-th layer, $\bar{Q}^m$ is the power density of the internal heat sources in the $m$-th layer, $\bar{\lambda}_{ij}^m$ are the thermal conductivities of the material of the $m$-th layer (in general all these quantities are functions of all the spatial variables), $\bar{H}_m = \text{const} > 0$ is the y coordinate of the boundary between the $m$-th and $(m+1)$-layers ($\bar{H}_m \equiv 0$, $\bar{H}_m = \bar{H}$), and $\bar{G}$ is the region occupied by the plate in plan. Here and henceforth dimensional functions and quantities will be given a bar, while the dimensionless functions and quantities corresponding to them will be denoted by the same symbols without a bar.

On the contact surfaces $\bar{x}_3 = \bar{H}_m$ of the $m$-th and $(m+1)$-layers the following conditions for the heat-flux and temperature solutions to be matched must be satisfied

$$\bar{x}_3 = \bar{H}_m: \ \bar{q}_3^{m+1} = \bar{q}_3^m, \quad \bar{T}^{m+1} = \bar{T}^m; \quad 1 \le m \le M-1 \tag{1.2}$$

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The following boundary conditions of general form are specified on the faces of the plate ($\bar{x}_3 = 0, \bar{H}$)

$$\bar{x}_3 = 0: -\beta^-\bar{q}_3^1 = \gamma^-\bar{Q}^-(\bar{x}_1, \bar{x}_2) + \delta^-\bar{\alpha}^-(\bar{T}^1(\bar{x}_1, \bar{x}_2, 0) - \bar{T}_\infty^-)$$
$$\bar{x}_3 = \bar{H}: \beta^+\bar{q}_3^M = \gamma^+\bar{Q}^+(\bar{x}_1, \bar{x}_2) + \delta^+\bar{\alpha}^+(\bar{T}^M(\bar{x}_1, \bar{x}_2, \bar{H}) - \bar{T}_\infty^+) \quad (1.3)$$

The superscript plus corresponds to the upper face of the plate, while the superscript minus corresponds to the lower face of the plate, $\bar{Q}^\pm$ are the projections of the heat-flux vector onto the direction of the outward normal, specified on the faces, $\bar{\alpha}^\pm$ are the coefficients of convective heat exchange with the environment on the sides of the upper and lower faces, $\bar{T}_\infty^\pm$ is the temperature of the environment on the sides of the upper and lower faces, and $\beta^\pm$, $\gamma^\pm$ and $\delta^\pm$ are switching functions, which enable any type of boundary conditions on the faces to be specified.

The following boundary conditions, similar to (1.3), are also specified on the end face of the plate

$$(\bar{x}_1, \bar{x}_2) \in \bar{\Gamma}, \quad \bar{H}_{m-1} \le \bar{x}_3 \le \bar{H}_m: \beta\sum_{i=1}^{2} n_i\bar{q}_i^m =$$
$$\gamma\bar{q}^m + \delta\bar{\alpha}^m(\bar{T}^m - \bar{T}_\infty); \quad 1 \le m \le M \quad (1.4)$$

where $n_i$ are the components of the vector of the unit normal to the end face of the plate ($n_3 = 0$), $\bar{q}^m$ is the specified heat flux through the end face of the $m$-th layer, $\bar{\alpha}^m$ is the Newton heat-exchange coefficient between the $m$-th layer and the environment on the end-face side, $\bar{T}_\infty$ is the temperature of the environment on the end-face side, $\beta$, $\gamma$ and $\delta$ are the switching functions, which enable any type of boundary conditions on the end face to specified, and $\bar{\Gamma}$ is the contour which bounds the region $\bar{G}$.

We will introduce the following dimensionless independent variables and small parameter

$$x_i = \frac{\bar{x}_i}{a}, \quad i = 1, 2, \quad x_3 = \frac{\bar{x}_3}{\bar{H}} \ (0 \le x_3 \le 1), \quad \varepsilon = \frac{\bar{H}}{a} \quad (1.5)$$

and also the following notation

$$T^m = \frac{\bar{T}^m}{\bar{T}_*}, \quad \lambda_{ij}^m = \frac{\bar{\lambda}_{ij}^m}{\bar{\lambda}_*}, \quad Q^m = \frac{\bar{Q}^m a^2}{\bar{\lambda}_*\bar{T}_*},$$
$$T_\infty^\pm = \frac{\bar{T}_\infty^\pm}{\bar{T}_*}, \quad Q^\pm = \frac{\bar{Q}^\pm a}{\bar{\lambda}_*\bar{T}_*}$$
$$\alpha^\pm = \frac{\bar{\alpha}^\pm a}{\bar{\lambda}_*}, \quad T_\infty = \frac{\bar{T}_\infty}{\bar{T}_*}, \quad q^m = \frac{\bar{q}^m a}{\bar{\lambda}_*\bar{T}_*},$$
$$\alpha^m = \frac{\bar{\alpha}^m a}{\bar{\lambda}_*}, \quad H_m = \frac{\bar{H}_m}{\bar{H}}$$
$$H_0 = 0, \quad H_M = H = \frac{\bar{H}}{\bar{H}} = 1;$$
$$i, j = 1, 2, 3; \quad 0 \le m \le M \quad (1.6)$$

where $a$ is the characteristic dimension of the region $\bar{G}$, $\bar{T}_*$ is a certain characteristic value of the temperature of the structure (for example, the temperature of the natural state), and $\bar{\lambda}_*$ is the characteristic value of the thermal conductivity of the materials of the layers of the plate (for example, the maximum over the layers of the greatest of the principal values of the thermal-conductivity tensor $\bar{\lambda}_{ij}^m$).

The heat conduction problem can be written in dimensionless form as follows (the subscript after the comma denotes partial differentiation with respect to the corresponding variable $x_i$ ($i = 1, 2, 3$)):

$$\varepsilon^2 L_2^m(T^m) + \varepsilon L_1^m(T^m) + (\lambda_{33}^m T_{,3}^m)_{,3} = -\varepsilon^2 Q^m(\mathbf{x}),$$
$$\mathbf{x} = \{x_1, x_2, x_3\} \quad (1.7)$$

$$L_1^m(T^m) \equiv (\lambda_{13}^m T_{,3}^m)_{,1} + (\lambda_{23}^m T_{,3}^m)_{,2} + (\Lambda_3^m(T^m))_{,3}$$
$$L_2^m(T^m) \equiv (\Lambda_1^m(T^m))_{,1} + (\Lambda_2^m(T^m))_{,2},$$
$$\Lambda_i^m(\cdot) = \lambda_{i1}^m(\cdot)_{,1} + \lambda_{i2}^m(\cdot)_{,2}$$
$$(x_1, x_2) \in G, \quad x_3 = H_m: \varepsilon\Lambda_3^{m+1}(T^{m+1}) + \lambda_{33}^{m+1}T_{,3}^{m+1} =$$
$$\varepsilon\Lambda_3^m(T^m) + \lambda_{33}^m T_{,3}^m \quad T^{m+1} = T^m; \quad 1 \le m \le M-1 \quad (1.8)$$

$$(x_1, x_2) \in G, \quad x_3 = 0: \beta^-[\varepsilon\Lambda_3^1(T^1) + \lambda_{33}^1 T_{,3}^1] =$$
$$\varepsilon\gamma^- Q^- + \varepsilon\delta^-\alpha^-(T^1 - T_\infty^-)$$
$$(x_1, x_2) \in G, \quad x_3 = H: -\beta^+[\varepsilon\Lambda_3^M(T^M) + \lambda_{33}^M T_{,3}^M] =$$
$$\varepsilon\gamma^+ Q^+ + \varepsilon\delta^+\alpha^+(T^M - T_\infty^+) \quad (1.9)$$

$$(x_1, x_2) \in \Gamma, \quad H_{m-1} \le x_3 \le H_m: -\beta\varepsilon\sum_{i=1}^{2} n_i\sum_{j=1}^{2}\lambda_{ij}^m T_{,j}^m -$$
$$-\beta(n_1\lambda_{13}^m + n_2\lambda_{23}^m)T_{,3}^m = \varepsilon\gamma q^m + \varepsilon\delta\alpha^m(T^m - T_\infty);$$
$$1 \le m \le M \quad (1.10)$$

As usual, we will assume that the temperature $\bar{T}$ differs only slightly from the value $\bar{T}_*$ (otherwise it would be necessary to take into account the thermal sensitivity of the materials of the plate layers, which is outside the framework of the present investigation). If we assume that a change in the plate thickness $\bar{H}$ corresponds to a change in the small parameter $\varepsilon$ (the values of $\bar{H}_m$ then vary in proportion to the change in $\bar{H}$, i.e. $H_m = \bar{H}_m/\bar{H} = \text{const}$) for the geometry of the structure in plan (for a fixed characteristic dimension $a$), then as $\varepsilon \to 0$ the order of the quantities $\lambda_{ij}^m, Q^m, T_\infty^\pm, \alpha^\pm, T_\infty, q^m, \alpha_m (i, j = 1, 2, 3, 1 \le m)$ is equal to unity.

The presence of the small parameter $\varepsilon$ in higher derivatives in problem (1.7)–(1.10) indicates that the problem is singularly perturbed, and hence its solution will be investigated in the form

$$T^m = T_*^m + T_b^m, \quad 1 \le m \le M \quad (1.11)$$

where $T_*^m$ is the main temperature field in the $m$-th layer and $T_b^m$ are the corrections to the main temperature field in the boundary layer in the neighbourhood of the end face of the plate.

Further, we will construct the main temperature field $T_*^m$ in the plate by the method of asymptotic analysis. Different asymptotic expansions must be used for different boundary conditions on the faces (1.9).

## 2. The case when the heat fluxes are specified on the faces

If only the heat fluxes ($\beta^{\pm}=1, \gamma^{\pm}=1, \delta^{\pm}=0$) are specified on both faces, we will choose as the outer expansion

$$T_*^m(\mathbf{x}) \sim \frac{1}{\varepsilon}\sum_{k=0}^{\infty} T_k^m(\mathbf{x})\varepsilon^k \tag{2.1}$$

After substituting expansion (2.1) into Eq. (1.7) and conditions (1.8)–(1.10) we collect terms in like powers of $\varepsilon$ in the relations obtained. This gives a chain of inequalities for determining the functions $T_k^m(\mathbf{x})$

$$(x_1,x_2)\in G,\quad H_{m-1}\le x_3\le H_m:\ (\lambda_{33}^m T_{0,3}^m)_{,3}=0;\quad 1\le m\le M \tag{2.2}$$

$$(x_1,x_2)\in G,\quad x_3=H_m:\ \lambda_{33}^{m+1}T_{0,3}^{m+1}=\lambda_{33}^m T_{0,3}^m,\quad T_0^{m+1}=T_0^m;\quad 1\le m\le M-1 \tag{2.3}$$

$$(x_1,x_2)\in G,\quad x_3=0:\ \beta^-\lambda_{33}^1 T_{0,3}^1=0$$
$$(x_1,x_2)\in G,\quad x_3=H:\ -\beta^+\lambda_{33}^M T_{0,3}^M=0 \tag{2.4}$$

$$(x_1,x_2)\in\Gamma,\quad H_{m-1}\le x_3\le H_m:\ -\beta(n_1\lambda_{13}^m+n_2\lambda_{23}^m)T_{0,3}^m=0;\quad 1\le m\le M \tag{2.5}$$

$$(x_1,x_2)\in G,\quad H_{m-1}\le x_3\le H_m:\ (\lambda_{33}^m T_{k,3}^m)_{,3}+L_1^m(T_{k-1}^m)+(1-\delta_{1k})L_2^m(T_{k-2}^m)=-\delta_{3k}Q^m(\mathbf{x});\quad 1\le m\le M \tag{2.6}$$

$$(x_1,x_2)\in G,\quad x_3=H_m:\ q_k^{m+1}(\mathbf{x})=q_k^m(\mathbf{x}),\quad T_k^{m+1}(\mathbf{x})=T_k^m(\mathbf{x});\quad 1\le m\le M-1 \tag{2.7}$$

$$(x_1,x_2)\in G,\quad x_3=0:\ q_k^1(\mathbf{x})=\delta_{2k}Q^-$$
$$(x_1,x_2)\in G,\quad x_3=H:\ -q_k^M(\mathbf{x})=\delta_{2k}Q^+ \tag{2.8}$$

$$(x_1,x_2)\in\Gamma,\quad H_{m-1}\le x_3\le H_m:\ -\beta\sum_{i=1}^2 n_i\sum_{j=1}^2\lambda_{ij}^m T_{k-1,j}^m-\beta(n_1\lambda_{13}^m+n_2\lambda_{23}^m)T_{k,3}^m-\delta\alpha^m T_{k-1}^m=\delta_{2k}(\gamma q^m-\delta\alpha^m T_\infty);\quad 1\le m\le M,\quad k=1,2,\dots \tag{2.9}$$

where

$$q_k^m(\mathbf{x})\equiv\lambda_{33}^m T_{k,3}^m+\lambda_{31}^m T_{k-1,1}^m+\lambda_{32}^m T_{k-1,2}^m,\quad 1\le m\le M,\quad k=1,2,\dots \tag{2.10}$$

and $\delta_{ik}=(i=1,2,3;\ k=1,2,\dots)$ is the Kronecker delta; in the case considered $\beta^{\pm}=1$.

Integrating Eq. (2.2), taking conditions (2.3) and (2.4) into account and also the conditions $\lambda_{33}^m>0$ (by virtue of Onsager's postulate[3]) we obtain

$$T_0^m(\mathbf{x})=\theta_0(x_1,x_2),\quad 1\le m\le M \tag{2.11}$$

where $\theta_0$ is an arbitrary function, to be determined subsequently. It follows from (2.11) that boundary condition (2.5) on the end face of the plate is satisfied identically.

We will introduce the following notation

$$\Theta_k^m\equiv\frac{\lambda_{31}^m\theta_{k,1}+\lambda_{32}^m\theta_{k,2}}{\lambda_{33}^m},\quad 1\le m\le M,\quad k=0,1,\dots$$

When $k=1$, integrating Eq. (2.6) with respect to the variable $x_3$, and taking Eqs. (2.11), (2.7), (2.8) and (2.10) into account, we obtain

$$q_1^m(\mathbf{x})=0 \tag{2.12}$$

Hence,

$$T_1^m(\mathbf{x})=\theta_1(x_1,x_2)-F_1^m(\mathbf{x}),\quad 1\le m\le M \tag{2.13}$$

where

$$F_1^m(\mathbf{x})\equiv\int_{H_{m-1}}^{x_3}\Theta_0^m dx_3+\sum_{l=1}^{m-1}\int_{H_{l-1}}^{H_l}\Theta_0^l dx_3 \tag{2.14}$$

and $\theta_1(x_1,x_2)\equiv T_1^1(x_1,x_2,0)$ is an arbitrary function, to be determined.

We express the derivative $T_{1,3}^m$ from Eq. (2.12) and substitute it into condition (2.9) for $k=1$. This condition then takes the form

$$(x_1,x_2)\in\Gamma,\quad H_{m-1}\le x_3\le H_m:\ -\beta\sum_{i=1}^2 n_i\sum_{j=1}^2\lambda_{ij}^m\theta_{0,j}+\beta(n_1\lambda_{13}^m+n_2\lambda_{23}^m)\Theta_0^m-\delta\alpha^m\theta_0=0;\quad 1\le m\le M \tag{2.15}$$

Since the materials of the layers are assumed to be arbitrary (and, in general, also non-uniform in thickness), while the function $\theta_0$ is independent of the variable $x_3$, boundary condition (2.15) cannot be satisfied exactly at all points of the end face of the plate, and hence here and henceforth the boundary conditions on the end face of the plate (2.15) and (2.9) ($k=2,3,\dots$) will be satisfied in the integral sense (by integrating these equations over the plate thickness), which is a necessary and sufficient condition for the attenuation of the boundary layers.[1]

Integratng relation (2.15) over the plate thickness, we obtain the following boundary condition on the end face for the function $\theta_0$

$$(x_1,x_2)\in\Gamma:\ \beta\sum_{i=1}^2\theta_{0,i}\sum_{m=1}^M\int_{H_{m-1}}^{H_m}\left[-n_1\lambda_{1i}^m-n_2\lambda_{2i}^m+\frac{\lambda_{3i}^m}{\lambda_{33}^m}(n_1\lambda_{13}^m+n_2\lambda_{23}^m)\right]dx_3-\delta\theta_0\sum_{m=1}^M\int_{H_{m-1}}^{H_m}\alpha^m dx_3=0 \tag{2.16}$$

When $k=2$, Eq. (2.6), in which we express the derivative $T_{1,3}^m$ from Eq. (2.12), can be converted to the form

$$q_{2,3}^m(\mathbf{x})=\partial^m(\theta_0)$$

The differential operator $\partial^m(\cdot)$ is defined as (the variable $x_3$ is the parameter)

$$\partial^m(\cdot) \equiv \sum_{i=1}^{2}\left[\frac{\lambda_{i3}^m}{\lambda_{33}^m}\Lambda_3^m(\cdot) - \Lambda_i^m(\cdot)\right]_{,i} \quad (2.17)$$

Integrating the last equation with respect to $x_3$, and taking into account conditions (2.7) and the first condition of (2.8) ($k=2$), we obtain

$$q_2^m(\mathbf{x}) = Q_2^m(\mathbf{x}), \quad 1 \le m \le M \quad (2.18)$$

where

$$Q_2^m(\mathbf{x}) \equiv Q^-(x_1, x_2) + \int_{H_{m-1}}^{x_3} \partial^m(\theta_0)dx_3 + D^m(\theta_0) \quad (2.19)$$

The differential operator $D^m(\cdot)$ has the form

$$D^m(\cdot) \equiv \sum_{l=1}^{m-1}\int_{H_{l-1}}^{H_l}\partial^m(\cdot)dx_3, \quad D^1(\cdot) \equiv 0 \quad (2.20)$$

It follows from Eq. (2.18) when $m=\mathrm{M}$ and $x_3=H_M=H$ and from the second equation of (2.8) ($k=2$), taking relations (2.19) and (2.20) into account, that

$$(x_1, x_2) \in G\colon D^{M+1}(\theta_0) = -Q^-(x_1, x_2) - Q^+(x_1, x_2) \quad (2.21)$$

This equation defines the function $\theta_0(x_1, x_2)$ for boundary condition (2.16) on the contour $\Gamma$. Knowing the function $\theta_0$ from boundary-value problem (2.21), (2.16), we obtain, by virtue of the relations (2.14), (2.19) and (2.20), the known right-hand side of Eq. (2.18) and the known function $F_1^m(\mathbf{x})$ in equality (2.13).

We will introduce the following notation

$$f_k^m \equiv (Q_k^m(\mathbf{x}) + \Lambda_3^m(F_{k-1}^m))/\lambda_{33}^m, \quad 1 \le m \le M, \quad k = 2, 3, \ldots$$

Substituting expression (2.13) into Eq. (2.18) and taking Eq. (2.10) into account, we obtain

$$T_{2,3}^m = f_2^m - \Theta_1^m \quad (2.22)$$

(the first term on the right-hand side is a known function). Integrating this equation with respect to $x_3$, taking the second relation of (2.7) into account ($k=2$), we obtain

$$T_2^m(\mathbf{x}) = \theta_2(x_1, x_2) - F_2^m(\mathbf{x}), \quad 1 \le m \le M \quad (2.23)$$

where

$$F_2^m(\mathbf{x}) \equiv \int_{H_{m-1}}^{x_3}(\Theta_1^m - f_2^m)dx_3 + \sum_{l=1}^{m-1}\int_{H_{l-1}}^{H_l}(\Theta_1^l - f_2^l)dx_3 \quad (2.24)$$

and $\theta_2(x_1, x_2) \equiv T_2^1(x_1, x_2, 0)$ is an arbitrary function, to be determined.

We substitute expression (2.22) into the boundary condition on the end face (2.9) ($k=2$) and integrate it over the plate thickness, taking equality (2.13) into account. We obtain

$$\beta\sum_{i=1}^{2}\theta_{1,i}\sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}\left[-n_1\lambda_{1i}^m - n_2\lambda_{2i}^m + \frac{\lambda_{3i}^m}{\lambda_{33}^m}(n_1\lambda_{13}^m + n_2\lambda_{23}^m)\right]dx_3 - \delta\theta_1\sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}\alpha^m dx_3 =$$

$$= \sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}\left[-\beta\sum_{i=1}^{2}n_i\Lambda_i^m(F_1^m) + \beta(n_1\lambda_{13}^m + n_2\lambda_{23}^m)f_2^m + \gamma q^m - \delta\alpha^m(F_1^m + T_\infty)\right]dx_3 \quad (2.25)$$

In view of the formal similarity of relations (2.6)–(2.9) for $k=2$ and $k\ge 3$ and Eqs. (2.23), (2.13) and (2.18), (2.12) we can construct a solution for boundary-value problem (2.6)–(2.9) for $k\ge 3$ by mathematical induction. Suppose, for a certain $k\ge 3$, the following assumptions hold

$$q_{k-1}^m(\mathbf{x}) = Q_{k-1}^m(\mathbf{x}), \quad 1 \le m \le M \quad (2.26)$$

$$T_{k-2}^m(\mathbf{x}) = \theta_{k-2}(x_1, x_2) - F_{k-2}^m(\mathbf{x}), \quad 1 \le m \le M \quad (2.27)$$

where $Q_{k-1}^m(\mathbf{x})$ and $F_{k-2}^m(\mathbf{x})$ are assumed to be already known functions. (When $k=3$ these assumptions hold since Eqs. (2.18) and (2.13) hold and the functions $Q_2^m$ and $F_1^m$ are known from relations (2.19) and (2.14) and the solved boundary-value problem (2.16), (2.21).) We will show that, for the next value of k, the structure of the solution is similar to Eqs. (2.26) and (2.27).

We express the derivative $T_{k-1,3}^m$ from Eq. (2.26) and substitute it into Eq. (2.6). Then, after using Eq. (2.27) and taking relation (2.10) into account we obtain

$$q_{k,3}^m = -\delta_{3k}Q^m(\mathbf{x}) - \sum_{i=1}^{2}\left(\frac{\lambda_{i3}^m}{\lambda_{33}^m}Q_{k-1}^m\right)_{,i} - \partial^m(F_{k-2}^m) + \partial^m(\theta_{k-2})$$

Integrating this equation with respect to $x_3$, taking Eqs. (2.7), (2.8) and (2.10) into account we will have

$$q_k^m(\mathbf{x}) = Q_k^m(\mathbf{x}), \quad 1 \le m \le M \quad (2.28)$$

where

$$Q_k^m(\mathbf{x}) \equiv -\int_{H_{m-1}}^{x_3}\left[\delta_{3k}Q^m(\mathbf{x}) + \sum_{i=1}^{2}\left(\frac{\lambda_{i3}^m}{\lambda_{33}^m}Q_{k-1}^m\right)_{,i} + \partial^m(F_{k-2}^m) - \partial^m(\theta_{k-2})\right]dx_3 -$$

$$-\sum_{l=1}^{m-1}\int_{H_{l-1}}^{H_l}\left[\delta_{3k}Q^l(\mathbf{x}) + \sum_{i=1}^{2}\left(\frac{\lambda_{i3}^l}{\lambda_{33}^l}Q_{k-1}^l\right)_{,i}\right]dx_3 - D^m(F_{k-2}^m) + D^m(\theta_{k-2}) \quad (2.29)$$

The differential operators $\partial^m(\cdot)$ and $D^m(\cdot)$ are defined by formulae (2.17) and (2.20).

From relations (2.28) with $m=M$ and $x_3=H_M=H$ and from the second condition of (2.8), taking definition (2.20) into account, it follows that

$$(x_1, x_2) \in G{:}\ D^{M+1}(\theta_{k-2}) = \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \left[\delta_{3k} Q^m(\mathbf{x}) + \sum_{i=1}^{2} \left(\frac{\lambda_{i3}^m}{\lambda_{33}^m} Q_{k-1}^m\right)_{,i} + \partial^m(F_{k-2}^m)\right] dx_3 \tag{2.30}$$

where the right-hand side is a known function of the variables $x_1$ and $x_2$, since the functions $Q_{k-1}^m$ and $F_{k-2}^m$ are assumed to be already known (see (2.26) and (2.27)).

When $k = 3$ Eq. (2.30) defines the function $\theta_1(x_1, x_2)$ for boundary condition (2.25), specified on the contour $\Gamma$. When $k \geq 4$ we obtain the boundary condition for Eq. (2.30) by integrating Eq. (2.9) over the plate thickness for the previous value of k (replacing k by k-1 in (2.9)), then, eliminating the derivative $T_{k-1,3}^m$ from (2.9) using (2.26) and using Eq. (2.27), we will have

$$\beta \sum_{i=1}^{2} \theta_{k-2,i} \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \left[-n_1\lambda_{1i}^m - n_2\lambda_{2i}^m + \frac{\lambda_{3i}^m}{\lambda_{33}^n}(n_1\lambda_{13}^m + n_2\lambda_{23}^m)\right] dx_3 - \delta\theta_{k-2} \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \alpha^m dx_3 =$$
$$= \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \left[-\beta \sum_{i=1}^{2} n_i \Lambda_i^m(F_{k-2}^m) + \beta(n_1\lambda_{13}^m + n_2\lambda_{23}^m) f_{k-1}^m - \delta\alpha^m F_{k-2}^m\right] dx_3, \quad k = 4, 5, 6, \ldots \tag{2.31}$$

We express the derivative $T_{k-1,3}^m$ from Eq. (2.26) and integrate the equation obtained with respect to $x_3$. Then, taking relation (2.27) into account we obtain

$$T_{k-1}^m(\mathbf{x}) = \theta_{k-1}(x_1, x_2) - F_{k-1}^m(\mathbf{x}), \quad 1 \leq m \leq M \tag{2.32}$$

where

$$F_{k-1}^m(\mathbf{x}) \equiv \int_{H_{m-1}}^{x_3} (\Theta_{k-2}^m - f_{k-1}^m) dx_3 + \sum_{l=1}^{m-1} \int_{H_{l-1}}^{H_l} (\Theta_{k-2}^l - f_{k-1}^l) dx_3 \tag{2.33}$$

and $\theta_{k-1}(x_1, x_2) \equiv T_{k-1}^1(x_1, x_2, 0)$ is an arbitrary function, to be determined. (When $k = 3$ Eqs. (2.32) and (2.33) are identical with (2.23) and (2.24) respectively).

By determining the function $\theta_{k-2}$ from boundary-value problem (2.30), (2.25) (for $k = 3$) or (2.30), (2.31) (for $k \geq 4$), we will have, by virtue of relations (2.33) and (2.29) and assumptions (2.26) and (2.27), the known functions $F_{k-1}^m$ and $Q_k^m$ in Eqs. (2.28) and (2.32), which formally are completely identical with (2.26) and (2.7). Hence, assumptions (2.26) and (2.27) also remain true for the next value of k, and hence, using the scheme (2.26)–(2.33) we can construct a solution of problem (2.6)–(2.9) for a new value of $k$, etc.

The proposed algorithm for determining the basic three-dimensional temperature field in a laminated anisotropic plate shows that, to calculate the unknown coefficients $T_k^m$ in asymptotic expansion (2.1) for each $k = 0, 1, 2, \ldots$, it is necessary to integrate the two-dimensional equations (2.21) and (2.30), which differ solely in the known right-hand sides and take the form

$$(x_1, x_2) \in G{:}\ D^{M+1}(\theta_{k-2}) = W_k(x_1, x_2) \tag{2.34}$$

The function $W_k(x_1, x_2)$ is defined by the right-hand side of Eq. (2.21) for $k = 2$ and the right-hand side of Eq. (2.30) for $k \geq 3$.

In addition to Eq. (2.34), we will consider the following equation

$$(x_1, x_2) \in G, \quad H_{m-1} \leq x_3 \leq H_m{:}\ \partial^m(\theta_{k-2}) = w_k(\mathbf{x}), \quad 1 \leq m \leq M \tag{2.35}$$

where the variable $x_3$ serves as the parameter, while the differential operator on the left-hand side is identical with the integrand in formula (2.20). (In particular, for a single-layer plate ($M = 1$), the thermal conductivities of which are independent of the variable $x_3$, the left-hand side of Eq. (2.34), after dividing by H, reduces to the left-hand side of Eq. (2.35).) The characteristic equation for Eq. (2.35) has the form

$$\hat{\Lambda}_{22}^m x_2'^2 - 2\hat{\Lambda}_{12}^m x_2' + \hat{\Lambda}_{11}^m = 0;$$
$$\hat{\Lambda}_{rs}^m = \lambda_{rs}^m - \lambda_{r3}^m \lambda_{s3}^m / \lambda_{33}^m, \quad r, s = 1, 2 \tag{2.36}$$

where $x_2' = dx_2/dx_1$ is a derivative which specifies the direction of the characteristic for fixed $x_3$. The discriminant of this equation is

$$D = -4\det(\lambda_{ij}^m)/\lambda_{33}^m, \quad i, j = 1, 2, 3$$

where $\det(\lambda_{ij}^m)$ is the determinant of the thermal conductivity matrix. According to Onsager's postulate,[3] $\lambda_{33}^m > 0$, $\det(\lambda_{ij}^m) > 0$, and hence $D < 0$. Consequently, Eq. (2.35) is elliptic, and the following inequality holds for the coefficients in Eq. (2.36) for any $x_3$

$$(\hat{\Lambda}_{12}^m)^2 < \hat{\Lambda}_{11}^m \hat{\Lambda}_{22}^m$$

We will integrate this inequality over the plate thickness and apply Bunyakovskii's inequality to the left-hand side. We obtain

$$(\Lambda_{12})^2 < \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \hat{\Lambda}_{11}^m \hat{\Lambda}_{22}^m dx_3 < \Lambda_{11}\Lambda_{22};$$
$$\Lambda_{rs} = \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \hat{\Lambda}_{rs}^m dx_3 \tag{2.37}$$

The last inequality in the chain is a consequence of the fact that, by virtue of Onsager's postulate, the factors $\hat{\Lambda}_{11}^m$ and $\hat{\Lambda}_{12}^m$ are positive for all $x_3$.

Using inequaity (2.37) we can determine the type of resolvent (2.34). The discriminant of its characteristic equation has the form

$$D = 4(\Lambda_{12})^2 - 4\Lambda_{11}\Lambda_{22} \tag{2.38}$$

Hence, taking inequality (2.37) into account we obtain $D < 0$. Consequently, the resolvents (2.34), (2.21) and (2.30) are second-order elliptic equations, which depend on the two variables $x_1$ and $x_2$.

We will discuss some properties of asymptotic expansion (2.1). Using expansion (2.1) and Eqs. (2.11), (2.13), (2.14), (2.23) and (2.24), we can assert that when the materials of the layers are of uniform thickness ($\lambda_{ij,3}^m = 0, i, j, = 1, 2, 3, \ \ 1 \leq m \leq M$), when only the heat fluxes on the faces of the plate outside the limits of the boundary layer are given, and which arise in the neighbourhood of the front faces, the temperature, with an accuracy of $O(\varepsilon)$ is distributed linearly over the thickness of each layer and is distributed piecewise-linearly over the thickness of the whole packet (here, in expansion (2.1) we must confine ourselves to the first two terms), and with an accuracy of $O(\varepsilon^3)$ the temperature over the thickness of each layer has a quadratic distribution and has a piecewise-quadratic distribution over the thickness of the whole laminated structure (in decomposition (2.1) we must retain three terms).

It follows from expansion (2.1) that when $Q^{\pm} \neq 0$

$$T_*^m(\mathbf{x}) = O(1/\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \tag{2.39}$$

i.e., when $\varepsilon$ is reduced the temperature $T_*^m$ in each layer increases in modulus without limit. This fact has a physical explanation, namely: a reduction in the thickness of the plate and the layers corresponds to a reduction in $\varepsilon$ when the remaining input data of the problem are fixed (the dimensions of the plate in plan, the power density of the internal heat sources, and the heat fluxes on the faces). Since the characteristic dimension of the plate $a$ and the heat fluxes on the faces $Q^{\pm}$ are fixed, the dimensionless inflow (outflow) of heat through the surfaces is fixed:

$$Q_* = -\iint_G (Q^+ + Q^-)dx_1 dx_2 \tag{2.40}$$

When the thickness of the plate and the layers are reduced (when $\varepsilon$ is reduced) the volume of the structure and the area of the faces of the plate and the layers are reduced, and hence, in order to ensure a fixed outflow (inflow) of heat through these surfaces, equal to the value (2.4), when $\varepsilon$ decreases one must increase the modulus of the components of the heat flux in the plate, which lie in the plane of the structure, and of course, by Fourier's law, the modulus of the temperature gradient must increase without limit. A consequence of this will be an unlimited increase the modulus of the temperature as $\varepsilon \to 0$, which relation (2.39) also reflects.

It follows from expansion (2.1) and Eqs. (2.21) and (2.30) (when $k$ = 3), that the contribution to the temperature from the heat fluxes $Q^{\pm}$, specified on the faces of the plate, with respect to $\varepsilon$, is an order of magnitude greater than the contribution from the internal heat sources $Q^m$, since $Q^{\pm}$ define the function $T_0^m$ (via $\theta_0$) and subsequent $T_k^m$ ($k$ = 2, 3, ..., see Eq. (2.21), while $Q^m$ specifies the function $T_1^m$ (via $\theta_1$) and subsequent $T_k^m$ ($k$ = 2, 3, ..., see Eq. (2.30)). This fact also has a physical explanation. When $\varepsilon$ is reduced the volume of the plate and layers is also reduced, and of course, by virtue of the fact that $Q^m = O(1)$ when $\varepsilon \to 0$ the quantity of heat $Q_V$ produced in unit volume over the whole plate by the internal heat sources $Q^m$ is reduced, and $Q_V \to 0$ when $\varepsilon \to 0$, since the volume of the plate approaches zero. The quantity of heat $Q_*$ supplies to the structure in unit time due to the heat fluxes $Q^{\pm}$ is fixed and is independent of $\varepsilon$ (see Eq. (2.40)), and hence the heat fluxes $Q^{\pm}$, specified on the faces, turn out to have a greater influence on the temperature than the internal heat sources.

If the faces of the plate are thermally insulated ($Q^{\pm}$ = 0), we obtain from Eqs. (2.13), (2.16), (2.21), (2.11), (2.19) and (2.14)

$$T_0^m \equiv \theta_0 \equiv 0, \quad Q_2^m \equiv 0, \quad F_1^m \equiv 0,$$
$$T_1^m \equiv \theta_1(x_1, x_2), \quad 1 \le m \le M \tag{2.41}$$

but, when there are internal heat sources ($Q^m \neq 0$) it follows from Eqs. (2.13), (2.24), (2.25) and (2.30) (when $k$ = 3) in the general case that

$$F_2^m(\mathbf{x}) \neq 0, \quad 1 \le m \le M \tag{2.42}$$

It follows from relations (2.1), (2.23), (2.41) and (2.42) that in the case when the materials of the layers are of uniform thickness $\lambda_{ij,3}^m = 0, i, j, = 1, 2, 3, \quad 1 \le m \le M$, with an accuracy of $O(\varepsilon^2)$ we can regard the temperature as being distributed linearly, but not constant, over the thickness of each layer (piecewise-linear over the thickness of the packet of layers); with an accuracy of $O(\varepsilon)$ we can regard the temperature as being constant over the thickness of the layers and the plate outside the limits of the boundary layer.

If, at each point of each layer of the plate, one of the principal axes of anisotropy coincides with the $x_3$ direction, we have $\lambda_{31}^m = \lambda_{32}^m = 0, \quad 1 \le m \le M$. (Plates, the layers of which are reinforced in planes parallel to the plane considered $x_3$ = 0, for example, possess such properties.) In this case, when the faces are thermally insulated ($Q^{\pm}$ = 0) and there are internal heat sources ($Q^m \neq 0$) we obtain from relations (2.24), (2.14) and (2.19)

$$F_1^m(\mathbf{x}) \equiv F_2^m(\mathbf{x}) \equiv 0, \quad Q_2^m(\mathbf{x}) \equiv 0, \quad 1 \le m \le M \tag{2.43}$$

Consequently, when $\lambda_{31}^m = \lambda_{32}^m = 0$ and $Q^{\pm}$ = 0, it follows from Eqs. (2.1), (2.11), (2.13), (2.23) and (2.43) that, with an accuracy of $O(\varepsilon)$, the temperature can be regarded as constant over the plate thickness outside the limits of the boundary layer even when the materials of the layers are of non-uniform thickness ($\lambda_{ij,3}^m \neq 0$).

If the faces are not thermally insulated $Q^{\pm} \neq 0$, and the internal heat sources are uniformly distributed over the thickness of the layers ($Q_3^m = 0$), then, when $\lambda_{31}^m = \lambda_{32}^m = 0, \quad 1 \le m \le M$, by virtue of relations (2.29), (2.32), (2.33), (2.14), (2.19) and (2.24) it follows from expansion (2.1) that, with an accuracy of $O(\varepsilon)$, the temperature distribution over the thickness of the layer, outside the limits of the boundary layer, has a quadratic form (a piecewise-quadratic form over the thickness of the laminated plate as a whole).

It was shown in Ref. 2 that, for a single-layer plate, uniform over the thickness, when there are no internal heat sources, with thermal insulation of the faces and $\lambda_{31} = \lambda_{32} = 0$ outside the limits of the boundary layer, the temperature in the plate is constant over the thickness. The same result is obtained for a single-layer ($M$ = 1) plate from relations (2.11)–(2.33) when

$$\lambda_{31}^1 = \lambda_{32}^1 = 0, \quad \lambda_{ij,3}^1 = 0, \quad Q^1 \equiv 0, \quad Q^{\pm} \equiv 0$$

since $F_k^1 \equiv 0$ and $T_k^1 \equiv \theta_k(x_1, x_2) \ (k = 0, 1, 2, \ldots)$.

In the case of a laminated plate ($M \ge 2$) with $\lambda_{31}^m = \lambda_{32}^m = 0, \lambda_{ij,3}^m = 0 \ (i, j, = 1, 2, 3)$, thermal insulation of the faces $Q^{\pm} \equiv 0$ and no internal heat sources ($Q^m \equiv 0, 1 \le m \le M$), the temperature outside the limits of the boundary layer cannot be assumed to be constant over the thickness of the structure. This is due to the non-uniformity of the material over the thickness of the whole packet of layers and follows directly from relations (2.29), (2.32), (2.33) and (2.43), whence it follows that the functions $Q_3^m$, $F_3^m$ and $T_3^m$ depend on the transverse coordinate $x_3$. Hence, in a laminated plate with $Q^m \equiv 0, Q^{\pm} \equiv 0$ the temperature can only be constant over the thickness when there is also thermal insulation of the end faces of the structure ($q^m \equiv 0, \beta = \gamma = 1, \delta = 0$ under conditions (1.10)) or when a constant temperature ($T^m = T_\infty$ = const, $\beta = \gamma = 0, \delta\alpha^m = 1$ under conditions (1.10)) is specified on the end face; however, in these cases since the solution of boundary-value (1.7)–(1.10) is unique, the temperature everywhere in the plate is constant ($T^m(\mathbf{x}) = T_\infty$ = const, $1 \le m \le M$).

## 3. The case of heat exchange due to convection

If heat exchange occurs on the faces of the plate due to convection (possibly with a simultaneous inflow (outflow) of heat due to specified heat fluxes), i.e. boundary conditions of the general form (1.9) occur, where the conditions $\delta^+ = 0$ or $\delta^- = 0$ are allowed (but not the simultaneous equality $\delta^+ = \delta^- = 0$, which corresponds to the case of boundary conditions with respect to the heat flux, considered in Section 2, then we choose as the outer asymptotic expansion of the temperature, unlike expansion (2.1),

$$T_*^m(\mathbf{x}) \sim \sum_{k=0}^{\infty} T_k^m(\mathbf{x})\varepsilon^k, \quad 1 \le m \le M \tag{3.1}$$

After substituting expansion (3.1) into Eq. (1.7) and conditions (1.8)–(1.10), we collect terms of like powers of $\varepsilon$ in the relations obtained. This gives a chain of equalities for determining the functions $T_k^m(\mathbf{x})$, where relations (2.2)–(2.5) and (2.7) remain true (for $k \ge 1$), to which we must add the equations

$$\begin{aligned}
&(x_1, x_2) \in G,\ x_3 = 0{:}\ \beta^- q_k^1(\mathbf{x}) - \delta^-\alpha^- T_{k-1}^1 = \\
&\quad \delta_{1k}(\gamma^- Q^- - \delta^-\alpha^- T_\infty^-) \\
&(x_1, x_2) \in G,\ x_3 = H{:}\ -\beta^+ q_k^M(\mathbf{x}) - \delta^+\alpha^+ T_{k-1}^M = \\
&\quad \delta_{1k}(\gamma^+ Q^+ - \delta^+\alpha^+ T_\infty^+)
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&(x_1, x_2) \in \Gamma, \quad H_{m-1} \le x_3 \le H_m{:}\ -\beta \sum_{i=1}^{2} n_i \sum_{j=1}^{2} \lambda_{ij}^m T_{k-1,\,j}^m - \\
&-\beta(n_1\lambda_{13}^m + n_2\lambda_{23}^m)T_{k,\,3}^m - \delta\alpha^m T_{k-1}^m = \delta_{1k}(\gamma q^m - \delta\alpha^m T_\infty); \\
&1 \le m \le M
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&(x_1, x_2) \in G, \quad H_{m-1} \le x_3 \le H_m{:}\ (\lambda_{33}^m T_{k,\,3}^m)_{,3} + L_1^m(T_{k-1}^m) + \\
&+(1-\delta_{1k})L_2^m(T_{k-2}^m) = -\delta_{2k}Q^m(\mathbf{x}); \quad 1 \le m \le M \\
&k = 1, 2, \ldots
\end{aligned} \tag{3.4}$$

where we must take relation (2.10) into account.

We will construct a solution of this system, assuming that the equalities $\beta^- = 0$ and $\beta^+ = 0$ are not satisfied simultaneously, i.e. boundary conditions in the form of known values of the temperature $T_\infty^\pm$ are not specified simultaneously on both faces. To fix our ideas, we will assume that $\beta^- = 1$ (convective heat transfer or the heat flux is specified on the lower face).

Integrating Eq. (2.2), taking conditions (2.3), (2.4) into account and assuming $\lambda_{33}^m > 0$, we obtain Eq. (2.11), from which it follows that condition (2.5) is satisfied identically on the face of the plate. Integrating Eq. (2.6) ($k = 1$) with respect to the variable $x_3$, and taking relations (2.11), (2.7) and (3.2) into account, we obtain

$$\lambda_{33}^m T_{1,\,3}^m + \lambda_{31}^m \theta_{0,\,1} + \lambda_{32}^m \theta_{0,\,2} = Q_1(x_1, x_2) \tag{3.5}$$

where

$$\begin{aligned}
&\theta_0(x_1, x_2) = -(\gamma^+ Q^+ + \beta^+\gamma^- Q^-)/\Delta + (\delta^+\alpha^+ T_\infty^+ + \beta^+\delta^-\alpha^- T_\infty^-)/\Delta \\
&Q_1 = [\delta^+\alpha^+(\gamma^- Q^- - \delta^-\alpha^- T_\infty^-) - \delta^-\alpha^-(\gamma^+ Q^+ - \delta^+\alpha^+ T_\infty^+)]/\Delta \\
&\Delta = \delta^+\alpha^+ + \beta^+\delta^-\alpha^-
\end{aligned} \tag{3.6}$$

Since the switching functions can only take the values $\beta^+ = 0$, $\delta^\pm = 0$ or $\beta^+ = 1$, $\delta^\pm = 1$ and $\beta^+$ and $\delta^+$ cannot be equal to zero simultaneously, and also, by virtue of the assumptions made above, the equalities $\delta^- = 0$ and $\delta^+ = 0$ cannot be satisfied simultaneously, the quantity $\Delta$ is non-zero, and the functions $\theta_0$ and $Q_1$ in relations (2.11) and (3.5) are uniquely defined by equalities (3.6).

Expressing the derivative $T_{1,3}^m$ from Eq. (3.5) and integrating the relation obtained with respect to $x_3$, we will have representation (2.13) or $T_1^m$, where

$$F_1^m(\mathbf{x}) \equiv \int_{H_{m-1}}^{x_3} \left(\Theta_0^m - \frac{Q_1}{\lambda_{33}^m}\right)dx_3 + \sum_{l=1}^{m-1} \int_{H_{l-1}}^{H_l} \left(\Theta_0^l - \frac{Q_1}{\lambda_{33}^l}\right)dx_3 \tag{3.7}$$

is a known function by virtue of Eqs. (3.6).

The boundary condition on the end face (3.3) ($k = 1$) after substituting the function $T_0^m = \theta_0$ from the first equality of (3.6) and the derivative $T_{1,3}^m$ from Eq. (3.5) into it, can be satisfied identically (both in the local and in the integral sense) only in exceptional cases. In general, the boundary condition on the end face of the plate can only be satisfied after considering the boundary layers.

We introduce the following notation

$$t_k^m = (Q_k^m - \Lambda_3^m(T_{k-1}^m))/\lambda_{33}^m, \quad 1 \le m \le M, \quad k = 2, 3, \ldots$$

and we will assume below that for any $k \ge 2$ the function $T_{k-2}^m$ is already known and equalities (2.26) and (2.32) hold, where the functions $Q_{k-1}^m(\mathbf{x})$ and $F_{k-1}^m(\mathbf{x})$ are also already known (for $k = 2$ these assumptions are justified in view of the fact that equalities (3.5)–(3.7), (2.11) and (2.13) are satisfied); then, taking relations (2.10) and (2.26) into account, Eq. (3.4) can be rewritten in the form

$$q_{k,\,3}^m = -\delta_{2k}Q^m(\mathbf{x}) - \sum_{i=1}^{2} (\lambda_{i3}^m t_{k-1}^m + \Lambda_i^m(T_{k-2}^m))_{,i}$$

where the right-hand side is known. Integrating this equation over the variable $x_3$ and taking relation (2.10) into account, we obtain

$$\begin{aligned}
&\lambda_{33}^m T_{k,\,3}^m + \lambda_{31}^m T_{k-1,\,1}^m + \lambda_{32}^m T_{k-1,\,2}^m = Q_k^m(\mathbf{x}), \\
&1 \le m \le M, \quad k \ge 2
\end{aligned} \tag{3.8}$$

where

$$Q_k^m(\mathbf{x}) \equiv Q_k^0(x_1, x_2) - \Phi_k^m(\mathbf{x}) \tag{3.9}$$

$$\begin{aligned}
\Phi_k^m(\mathbf{x}) \equiv &\int_{H_{m-1}}^{x_3} \left[\delta_{2k}Q^m(\mathbf{x}) + \sum_{i=1}^{2} (\lambda_{i3}^m t_{k-1}^m + \Lambda_i^m(T_{k-2}^m))_{,i}\right]dx_3 + \\
&+\sum_{l=1}^{m-1} \int_{H_{l-1}}^{H_l} \left[\delta_{2k}Q^l(\mathbf{x}) + \sum_{i=1}^{2} (\lambda_{i3}^l t_{k-1}^l + \Lambda_i^l(T_{k-2}^l))_{,i}\right]dx_3
\end{aligned} \tag{3.10}$$

$Q_k^0(x_1, x_2) \equiv Q_k^1(x_1, x_2, 0)$ is a function to be determined and $\Phi_k^m$ is a known function.

Substituting expressions (2.32), (3.8) and (3.9) into boundary conditions (3.2) ($k \ge 2$), we obtain ($\beta^- = 1$)

$$\begin{aligned}
&Q_k^0 - \delta^-\alpha^-\theta_{k-1} = 0; \\
&x_3 = H{:}\ -\beta^+(Q_k^0 - \Phi_k^M) - \delta^+\alpha^+(\theta_{k-1} - F_k^M) = 0
\end{aligned}$$

Hence

$$\begin{aligned}
&\theta_{k-1}(x_1, x_2) = \frac{1}{\Delta}(\Phi_k^M + \delta^+\alpha^+ F_{k-1}^M)\Big|_{x_3 = H}, \\
&Q_k^0(x_1, x_2) = \delta^-\alpha^-\theta_{k-1}
\end{aligned} \tag{3.11}$$

The denominators on the right-hand sides of Eqs. (3.6) and (3.11) are identical and non-zero, and hence the functions $\theta_{k-1}$ and $Q_k^0$ are uniquely defined by Eqs. (3.11), and in view of equalities (3.9), (3.10) and (2.32) and the assumption made above that the function $F_{k-1}^m$ is already known, we obtain the functions $T_{k-1}^m$ and $Q_k^m$ known in relations (3.8). Expressing the derivative $T_{k,3}^m$ from Eq. (3.8) and integrating the relation obtained over $x_3$, we will have

$$T_k^m(\mathbf{x}) = \theta_k(x_1, x_2) - F_k^m(\mathbf{x}), \quad 1 \le m \le M, \quad k \ge 2 \tag{3.12}$$

where

$$F_k^m(\mathbf{x}) = -\int_{H_{m-1}}^{x_3} t_k^m dx_3 - \sum_{l=1}^{m-1} \int_{H_{l-1}}^{H_l} t_k^l dx_3 \tag{3.13}$$

$\theta_k(x_1, x_2) \equiv T_k^1 \theta_k(x_1, x_2, 0)$ is a function to be determined and $F_k^m(\mathbf{x})$ is a known function.

By virtue of equalities (2.32) and (3.9)–(3.13) we obtain that the assumptions made above also hold for the next value of $k$, and hence, using the scheme (2.32) and (3.9)–(3.13) we can construct a solution of problem (3.2), (3.4) for the new value of $k$, etc.

When $k \ge 2$, as also for $k = 1$, boundary condition (3.3), after substituting into it the function $T_{k-1}^m(\mathbf{x})$ from Eq. (2.32) and the derivative $T_{k,3}^m$ from Eq. (3.8), taking equalities (3.13) and (3.9)–(3.11) into account, can be satisfied identically (both in the local and integral sense) only in exceptional cases. In the general case, the boundary conditions on the end faces of the plate $k \ge 2$ can only be satisfied after considering the boundary layers.

Hence, when at least on one of the faces of the plate (for example, the lowest one) the convective heat exchange is specified, all the unknown functions in expansion (3.1) are defined by the equalities (2.11), (2.13), (3.7), (3.12) and (3.13), in which the arbitrary functions are specified by the finite relations (3.6) and (3.11), and there is no need to determine them from the two-dimensional elliptic equations of the form (2.21), (2.30), (2.34), as was done in the case when only the heat fluxes $Q^\pm$ were specified on the faces.

It follows from relations (2.11), (3.6) and (3.9)–(3.12) that, in the case when boundary conditions of general form (1.9) are specified on the faces, like when only the heat fluxes are specified, the convective heat exchange and heat fluxes $Q^\pm$ on the faces of the plate have an order of magnitude, with respect to $\varepsilon$, greater effect on the temperature than internal heat sources, since the fluxes $Q\pm$ and the convective heat exchange determine the function $T_0^m$ (see relations (2.11) and (3.6)), and all the subsequent $T_1^m, T_2^m, \ldots$, while the power density of the internal heat sources $Q^m$ determine $T_1^m$ (see relations (2.13), (3.10) and (3.11) with $k = 2$) and all subsequent functions $T_2^m, T_3^m, \ldots$. This fact has the same physical explanation as in the case when only the heat fluxes $Q^\pm$ are specified on the faces.

It follows from relations (2.11), (2.13), (3.6), (3.7) and (3.9)–(3.13), when the materials of the layers are uniform over the thickness ($\lambda_{ij,3}^m = 0$), that, when the convective heat transfer on the faces is specified, the temperature outside the limits of the boundary layer can be assumed to have a linear distribution over the thickness of the layers with an accuracy of $O(\varepsilon^2)$ (a piecewise-linear distribution over the thickness of the whole packet); moreover, when the power density of the internal heat sources have a uniform distribution over the thickness of the layers ($Q_3^m = 0$), the temperature outside the limits of the boundary layer, with an accuracy of $O(\varepsilon^3)$, can be specified as having a quadratic distribution over the thickness of each layer (a piecewise-quadratic distribution over the thickness of the laminated plate).

An attempt to construct an asymptotic expansion of the solution of the problem of the heat conduction of a laminated plate for boundary conditions of general form (1.9) in the form (2.1) leads to the equality $T_0^m \equiv 0$, i.e., there are no terms of the order of $1/\varepsilon$ in expansion (2.1). Consequently, in the final analysis we arrive at an expansion of the form (3.1). The absence of a term $O(1/\varepsilon)$ in the asymptotic expansion of the temperature when there is convective heat transfer on the faces has a physical explanation. When $\varepsilon$ is reduced the thickness and the area of the end faces of the plate (and the layers) are reduced, and also the volume of the structure, and hence as $\varepsilon \to 0$ the inflow (outflow) of heat through the end face and the heat production due to the internal sources (sinks) are reduced in modulus. However, for a certain finite temperature a thermal balance is established between the inflow (outflow) of heat through the faces due to the heat fluxes $Q^\pm$ and convective heat transfer ($\delta^+ \ne 0$ and/or $\delta^- \ne 0$). This balance would be impossible if a term $O(1/\varepsilon)$ were present in the asymptotic expansion of the temperature, since in this case we would obtain $|T_*^m| \to \infty$ as $\varepsilon \to 0$ and the outflow (inflow) of heat due to convective heat transfer would be unlimitedly large in modulus for a limited and fixed inflow (outflow) of heat due to the fluxes $Q^\pm$ (see Eq. (2.40)).

In the boundary conditions of general form (1.9), the dimensionless coefficients $\alpha^\pm$, which characterize the Biot criterion,[1] may be of the order of unity, and may be large and small compared with unity. Thus, for a plate with copper outer layers ($\bar{\lambda}_{ii}^1 = \bar{\lambda}_{ii}^M = 400$ W/m K, Ref. 4) with a characteristic dimension in plan of $a = 1$ m for free convection of water ($\bar{\alpha}^\pm = 500$ W/m$^2$ K, Ref. 5) the dimensionless heat exchange coefficient $\alpha^\pm = 5/4$ (see formulae (1.6); for free convection of gases ($\bar{\alpha}^\pm = 30$ W/m$^2$ K, Ref. 5) for the same plate we obtain $\alpha^\pm = 3/40$; in the case of forced convection of water ($\bar{\alpha}^\pm = 10^4$ W/m$^2$ K, Ref. 5) for the same plate we have $\alpha^\pm = 25$, while for a plate with steel outer layers ($\bar{\lambda}_{ii}^1 = \bar{\lambda}_{ii}^M = 45$ W/m K, Ref. 4) of the same dimensions in plan $\alpha^\pm = 222$. Consequently, under certain conditions of heat transfer on the faces, the dimensionless quantities $\alpha^\pm$ can be considered to be small or large parameters independent of $\varepsilon$, and of course, instead of expansion (3.1) we can construct an asymptotic expansion of the temperature with respect to three independent parameters $\varepsilon$, $\alpha^+$, $\alpha^-$ (the corresponding calculations are not given here). However, the quantities $\alpha^\pm$ do not occur in the heat-conduction equation (1.7), and only determine the boundary conditions (1.9), and hence this expansion does not lead to any essential simplification of the two-dimensional resolvents for the coefficients of the asymptotic series, but generates a long chain of inequalities. The fundamental asymptotic properties determined by the parameters $\alpha^\pm$ can be followed using Eqs. (2.11), (2.13), (3.6), (3.7) and (3.9)–(3.13).

If the convective heat exchange ($\delta^\pm = 1$, $\beta^\pm = 1$) is specified on both faces, where at least one of them is intensive ($\alpha^+ \gg 1$ or $\alpha^- \gg 1$, or $\alpha^\pm \gg 1$) and $Q^\pm \ne 0$, the first term on the right-hand side of the first equation of (3.6) is a small quantity and approaches zero in the limiting case $\alpha^+ \to \infty$ or $\alpha^- \to \infty$, or $\alpha^\pm \to \infty$, i.e. by virtue of Eq. (2.11) the contribution to $T_0^m$ from the heat fluxes $Q^\pm$ is small, but it may have a considerable effect on the functions $Q_1$, $F_1^m$ and $T_1^m$ (see relations (3.6), (3.7) and (2.30)). The second term on the right-hand side of the first equation of (3.6) for any (small or large) values of $\alpha^\pm$ defines the mean value of the dimensionless temperatures of the surrounding medium above and below the plate, by the mixture rule. If there is intensive convective heat exchange $\alpha^\pm \ll 1$ on both faces, then when $Q^\pm \ne 0$ the first term on the right-hand side of the first equation of (3.6) may be large in modulus, which increases without limit in the limiting case $\alpha^\pm \to 0$. Then the function $T_0^m \equiv \theta_0$ is of the order of $1/\alpha^\pm$ as $\alpha^\pm \to 0$. If there is no heat flux, then $T_0^m = O(1)$ when $\alpha^\pm \to 0$ (see relations (3.6) and (2.11) when $Q^\pm = 0$). The functions $\theta_k$, $T_k^m$ possess similar asymptotic properties when $k \ge 1$ (see relations (3.9)–(3.13)).

If, to meet certain requirements, for example, a lifetime guarantee, it is necessary to maintain a fixed temperature $T_\infty^\pm$ on one of the faces of the plate, for example, the upper face ($x_3 = H$), while on the second face, for example, the lower face ($x_3 = 0$), the convective heat exchange ($\delta^+\alpha^+ = 1$, $\beta^+ = \gamma^+ = 0$) is specified, it follows from relations (3.6) and (2.11) that

$$T_0^m \equiv T_\infty^+(x_1, x_2), \quad \theta_1(x_1, x_2) = \gamma^- Q^- + \delta^-\alpha^-(T_\infty^+ - T_\infty^-)$$

Hence, we also obtain from relations (3.7) and (2.13) that, for $\delta^- \neq 0$ the function $T_1^m$ depends on $\alpha^-$. Similarly, a linear dependence of the functions $T_k^m$ on $\alpha^-$ when $k \geq 0$ follows from relations (3.9)–(3.13). Consequently, if intensive convective heat exchange ($\alpha^- \gg 1$, $\delta^- = 1$) is specified on the lower face, we obtain from relations (3.6), (3.7) and (3.9)–(3.13) that the function $T_k^m (k \geq 1)$ can have values of greater modulus, which increase without limit in the limiting case when $\alpha^- \to \infty$. For a small value of $\alpha^- \ll 1$ ($\delta^- = 1$) from the same equations we obtain $T_k^m \equiv O(1)$ when $\alpha^- \to 0$ ($k \geq 1$), i.e. all the coefficients $T_k^m$ of asymptotic series (3.1) remain bounded in modulus.

## 4. The case of boundary conditions of the first kind

If boundary conditions of the first kind (with respect to the temperature)

$$T_*^1(x_1, x_2, 0) = T_\infty^-, \quad T_*^M(x_1, x_2, H) = T_\infty^+$$

are specified on both faces, we must put $\beta^\pm = \gamma^\pm = 0$, $\delta^\pm\alpha^\pm = 1$ in boundary conditions (1.9). Then the asymptotic expansion must be constructed in the form (3.1), while in the chain of equalities (2.2)–(2.5), (2.7) and (3.2)–(3.4) we must use the following relations

$$(x_1, x_2) \in G, \quad x_3 = 0\colon T_0^1(\mathbf{x}) = T_\infty^-(x_1, x_2),$$
$$T_k^1(\mathbf{x}) = 0;$$
$$(x_1, x_2) \in G, \quad x_3 = H\colon T_0^M(\mathbf{x}) = T_\infty^+(x_1, x_2),$$
$$T_k^M(\mathbf{x}) = 0; \quad k \geq 1 \tag{4.1}$$

instead of (2.4) and (3.2).

Integrating Eq. (2.2), taking conditions (2.3) into account, we obtain

$$\lambda_{33}^m T_{0,3}^m = Q_0(x_1, x_2), \quad T_{0,3}^m = Q_0(x_1, x_2)/\lambda_{33}^m; \quad 1 \leq m \leq M \tag{4.2}$$

where $Q_0$ is a function to be determined. We will integrate the second equality of (4.2) taking conditions (2.3) and (4.1) into account. We obtain

$$T_0^m(\mathbf{x}) = T_\infty^-(x_1, x_2) + Q_0(x_1, x_2)\left(I^m(\mathbf{x}) + \sum_{l=1}^{m-1} I^l(x_1, x_2, H_l)\right) \tag{4.3}$$

$$Q_0(x_1, x_2) = (T_\infty^+ - T_\infty^-)\left[\sum_{m=1}^{M} I^m(x_1, x_2, H_m)\right]^{-1},$$
$$I^m(\mathbf{x}) = \int_{H_{m-1}}^{x_3} \frac{dx_3}{\lambda_{33}^m} \tag{4.4}$$

The function $T_0^m$ is completely defined by the finite relations (4.3) and (4.4), but unlike the previous case of convective heat exchange (Section 3), specified on the faces, here $T_0^m$ already depends on the transverse coordinate $x_3$.

Equation (2.6) ($k = 1$), taking relations (4.2) and (2.7) into account, can be integrated with respect to $x_3$, in which case we obtain equalities of the form (3.8) and (3.9), where

$$\Phi_1^m(\mathbf{x}) = I_1^m(\mathbf{x}) + \sum_{l=1}^{m-1} I_1^l(x_1, x_2, H_l),$$
$$I_1^m(\mathbf{x}) = \int_{H_{m-1}}^{x_3} \sum_{i=1}^{2} \left(\frac{\lambda_{i3}^m}{\lambda_{33}^m} Q_0\right)_{,i} dx_3 \tag{4.5}$$

We express the derivative $T_{1,3}^m$ from relation (3.8) when $k = 1$ and integrate the equation obtained with respect to the variable $x_3$, after which, taking conditions (2.7) and (4.1) and Eq. (3.9) into account, we determine

$$T_1^m(\mathbf{x}) = J_1^m(\mathbf{x}) + \sum_{l=1}^{m-1} J_1^l(x_1, x_2, H_l)$$
$$J_k^m(\mathbf{x}) \equiv \int_{H_{m-1}}^{x_3} \left(\frac{Q_k^0(x_1, x_2)}{\lambda_{33}^m} - \chi_k^m(\mathbf{x})\right) dx_3, \quad k = 1, 2, \ldots$$
$$\chi_k^m(\mathbf{x}) = (\Phi_k^m(\mathbf{x}) + \lambda_{31}^m T_{k-1,1}^m + \lambda_{32}^m T_{k-1,2}^m)/\lambda_{33}^m \tag{4.6}$$

where the function $Q_1^0$ is given by the equation

$$Q_1^0(x_1, x_2) = \left[\sum_{m=1}^{M} I^m(x_1, x_2, H_m)\right]^{-1} \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \chi_1^m(\mathbf{x}) dx_3 \tag{4.7}$$

Equalities (4.3)–(4.7) and (3.9) completely define the functions $T_1^m$ and $Q_1^m$.

Integrating Eq. (3.4) ($k = 2$) with respect to the variable $x_3$, taking conditions (2.7) into account, we obtain relations (3.8)–(3.10), where the function $\Phi_2^m$ is known by virtue of Eqs. (4.3)–(4.7). We express the derivative $T_{k,3}^m$ from relation (3.8) and integrate the equation obtained with respect to the variable $x_3$, after which, taking conditions (2.7) and (4.1) into account, we determine

$$T_k^m(\mathbf{x}) = J_k^m(\mathbf{x}) + \sum_{l=1}^{m-1} J_k^l(x_1, x_2, H_l), \quad k = 2, 3, \ldots \tag{4.8}$$

The function $Q_k^0$ is specified by an equation similar to (4.7)

$$Q_k^0(x_1, x_2) = \left[\sum_{m=1}^{M} I^m(x_1, x_2, H_m)\right]^{-1} \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \chi_k^m(\mathbf{x}) dx_3 \tag{4.9}$$

It follows from relations (3.9), (3.10), (4.8) and (4.9) that the functions $T_k^m$ and $Q_k^m$ are known, and hence, from the scheme (3.8)–(3.10), (4.8) and (4.9) we can determine the functions $T_k^m$ and $Q_k^m$ for a new value of $k$, etc.

Expansion (3.1), taking Eqs. (4.3)–(4.9) into account, possesses the same asymptotic properties (with respect to the parameter $\varepsilon$), as in the case, considered in Section 3, when the convective heat exchange on the faces of the plate is given. The only main difference is that, when the temperature $T_\infty^\pm$ is given on both faces, the first term (the function $T_0^m$) in expansion (3.1), acording to Eq. (4.3), depends on the transverse coordinate $x_3$. (When the convective heat exchange is given, according to Eq. (2.11), $T_0^m$ is independent of $x_3$.) Hence, it follows, in particular, that when the materials of the layers are uniform over the thickness ($\lambda_{ij,3}^m = 0$), by virtue of (4.3), with an accuracy of $O(\varepsilon)$, the temperature distribution in each layer in a transverse direction can be specified to be linear (piecewise-linear for the whole packet); with an accuracy of $O(\varepsilon^2)$, by virtue of relations (4.3)–(4.6), the temperature over the thickness of the layers can be specified to be quadratic (piecewise-quadratic for the whole plate).

It follows from Eqs. (4.3) and (4.4) that when the same temperature is set on both faces ($T_\infty^+ = T_\infty^-$), we obtain $T_0^m = T_\infty^+ = T_\infty^-$, $Q_0 = 0$, i.e. with an accuracy of $O(\varepsilon)$ we can assume the temperature to be constant over the thickness of the laminated plate; if, moreover, $T_\infty^+ = T_\infty^- = \text{const}$, then according to relations (4.3)–(4.7), we obtain $T_1^m = 0$, and the temperature over the thickness of the laminated packet, outside the limits of the boundary layer, can be assumed to be constant with an accuracy of $O(\varepsilon^2)$; if there are no internal heat sources ($Q_m \equiv 0$), from relations (4.8), (4.9) and (3.10) we additionally obtain $T_k^m \equiv 0 (k \geq 1)$, i.e. in this case, outside the limits of the boundary layer, the temperature in the plate is constant ($T_*^m = T_0^m = T_\infty^+ = T_\infty^- = \text{const}$).

The outer asymptotic expansions of the temperature constructed above can lead to discrepancies in boundary conditions (1.10) on the end faces of the plate,[6] to remove which one can use the usual procedure[1,6] of introducing in the neighbourhood of the contour $\Gamma$ inner "extended variables in the plane of the plate, corresponding to $x_1$ and $x_2$, and constructing an inner asymptotic expansion for the boundary layer with subsequent matching of this expansion with the outer expansion. An investigation of these problems is outside the scope of the present paper.

The outer asymptotic expansion obtained here and the estimates of the accuracy of the representation of the temperature by a linear or quadratic distribution over the thickness of the layers of the plate, based on them, can be used in calculations of the strength and pliability of thin-walled laminated structures, since the approximate theories of the bending of plates used in practice (Kirchhoff's, Timoshenko's, etc.[7]) only give acceptable accuracy outside the limits of the boundary layer.[7] Moreover, the asysmptotic expansion of the temperature constructed can also be employed in the asymptotic analysis of the thermoelastic behaviour of anisotropic plates. For example, when investigating the asymptotic properties of the solutions of uncoupled thermoelastic problems it was assumed,[8,9] that the temperature in the plate outside the limits of the boundary layer can be represented in the form (2.1), but this representation was not strictly justified, since the asymptotic properties of the solution of the heat-conduction problem were not investigated. Our investigation has shown that the main temperature field in an anisotropic plate in the most general case can be represented by expansion (2.1), which confirms the internal harmony of the uncoupled problem of thermoelasticity and the correctness of the asymptotic approach for solving this class of problems.

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